What is a (smooth) cubic surface?

**Definition**

zero set $S \subset \mathbb{P}^3$ of a homogeneous cubic polynomial $F(x, y, z, w)$

- **singular**
  \[x^2(x + w) = w(y^2 + z^2)\]

- **smooth**
  \[x^3 + y^3 + z^3 + w^3 = 0\]
The Cayley–Salmon Theorem

\[ M = \{ S \mid S \subset \mathbb{C}P^3 \text{ smooth cubic surface} \} = \{ F \mid F \text{ homogeneous smooth degree 3 in } \mathbb{C}[x,y,z,w] \}/\mathbb{C}^\times \]
The Cayley–Salmon Theorem

- $M_{\text{line}} = \{(S, L) \mid S \in M, L \subset S, L \text{ line}\}$

- $M = \{S \mid S \subset \mathbb{CP}^3 \text{ smooth cubic surface}\}$
  
  $= \{F \mid F \text{ homogeneous smooth degree 3 in } \mathbb{C}[x, y, z, w]\} / \mathbb{C}^\times$
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Theorem (Cayley–Salmon)

*The projection* \( M_{\text{line}} \rightarrow M \) *is a* 27 : 1 covering map.*
The lines on the Clebsch surface

Figure: 27 lines on the Clebsch surface: $x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$

Image credit: Greg Egan, via the AMS Visual Insight blog by John Baez
What about points?

- \( M_{\text{point}} = \{(S,p) \mid S \in M, \ p \in S\} \)

universal bundle

- \( M = \{S \mid S \subset \mathbb{CP}^3 \text{ smooth cubic surface}\} = \{F \mid F \text{ homogeneous smooth degree 3 in } \mathbb{C}[x,y,z,w]\} / \mathbb{C}^\times \)
From topology to point counts

- $M(\mathbb{F}_q)$: smooth cubic surfaces over $\mathbb{F}_q$
From topology to point counts

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- $M_\text{line}(\mathbb{F}_q)$: pairs $(S, L)$ over $\mathbb{F}_q$
From topology to point counts

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- $M_{\text{line}}(\mathbb{F}_q)$: pairs $(S, L)$ over $\mathbb{F}_q$

- Average number of lines: $\frac{\#M_{\text{line}}(\mathbb{F}_q)}{\#M(\mathbb{F}_q)}$
From topology to point counts

- $M(\mathbb{F}_q)$: smooth cubic surfaces over $\mathbb{F}_q$

- $M_{\text{line}}(\mathbb{F}_q)$: pairs $(S, L)$ over $\mathbb{F}_q$

- Average number of lines: $\frac{\#M_{\text{line}}(\mathbb{F}_q)}{\#M(\mathbb{F}_q)} = 1 + O\left(\frac{1}{\sqrt{q}}\right)$

- $M_{\text{line}}$ connected $\implies H^0(M) \cong H^0(M_{\text{line}})$
From topology to *point* counts

- $M(\mathbb{F}_q)$: smooth cubic surfaces over $\mathbb{F}_q$
- $M_{\text{point}}(\mathbb{F}_q)$: pairs $(S, p)$ over $\mathbb{F}_q$

Average number of points:

$$\# M_{\text{point}}(\mathbb{F}_q) = q^2 + O(q)$$

Need more knowledge about:

- $H^*(M)$
- $H^*(M_{\text{line}})$
- $H^*(M_{\text{point}})$
From topology to *point* counts

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- Need more knowledge about: $H^*(M), H^*(M_{\text{line}}), H^*(M_{\text{point}})$
Theorem (Vassiliev 1999)

\[ H^* (M; \mathbb{Q}) \cong H^* (\text{PGL}(4, \mathbb{C}); \mathbb{Q}). \]
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- Why \( \text{PGL}(4, \mathbb{C}) \)? Automorphism group of \( \mathbb{C}P^3 \)
Cohomology of $M$

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- **Why $\text{PGL}(4, \mathbb{C})$?** Automorphism group of $\mathbb{C}P^3$

- **Fix** $S_0 \in M \rightsquigarrow$ orbit map $g \mapsto g(S_0)$, $\text{PGL}(4, \mathbb{C}) \to M$
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- Fix $S_0 \in M \leadsto \text{orbit map } g \mapsto g(S_0), \text{PGL}(4, \mathbb{C}) \to M$

Theorem (Peters–Steenbrink 2003)

The orbit map induces $H^*(M; \mathbb{Q}) \cong H^*(\text{PGL}(4, \mathbb{C}); \mathbb{Q})$. 
Cohomology of $M_{\text{line}}$

**Theorem (D.)**

$H^*(M_{\text{line}}; \mathbb{Q}) \cong H^*(M; \mathbb{Q})$; induced by the covering map.

**Corollary**

Average number of lines $= 1$.

- In fact, $\# M_{\text{line}}(F_q) = \# M(F_q) = q^4 \cdot \# \text{PGL}(4, F_q)$. 

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arXiv:1803.04146
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*Average number of lines* $= 1$.

- In fact, $#M_{\text{line}}(\mathbb{F}_q) = #M(\mathbb{F}_q) = q^4(#\text{PGL}(4, \mathbb{F}_q))$. 
Theorem (D.)

\[ H^*(M_{\text{point}}; \mathbb{Q}) \cong H^*(M \times \mathbb{C}P^2; \mathbb{Q}). \]
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\[ H^* (M_{\text{point}}; \mathbb{Q}) \cong H^* (M \times \mathbb{C}P^2; \mathbb{Q}). \]

Corollary

Average number of points = \( q^2 + q + 1 \).
Improvement in progress: distribution of points

\[ t \quad 51840q^4 \times (\text{proportion of surfaces with } q^2 + tq + 1 \text{ points}) \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( 80q^4 + 240q^3 )</th>
<th>( - 400q - 240 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>( 3465q^4 - 1935q^3 + 2025q^2 - 8145q - 1890 )</td>
<td></td>
</tr>
<tr>
<td>(-1)</td>
<td>( 11664q^4 + 4320q^3 - 5184q^2 + 6480q + 4320 )</td>
<td></td>
</tr>
<tr>
<td>(0)</td>
<td>( 20820q^4 - 3060q^3 + 1620q^2 + 9660q - 720 )</td>
<td></td>
</tr>
<tr>
<td>(1)</td>
<td>( 13104q^4 - 720q^3 + 5184q^2 - 5040q - 2160 )</td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>( 2430q^4 + 1350q^3 - 4050q^2 - 2430q + 540 )</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>( 240q^4 )</td>
<td>( + 240q )</td>
</tr>
<tr>
<td>(4)</td>
<td>( 36q^4 - 180q^3 + 324q^2 - 180q )</td>
<td></td>
</tr>
<tr>
<td>(5)</td>
<td>( q^4 - 15q^3 + 81q^2 - 185q + 150 )</td>
<td></td>
</tr>
</tbody>
</table>

*using results from Bergvall–Gounelas ’19*
### Improvement in progress: other markings†

<table>
<thead>
<tr>
<th>marking</th>
<th>average count</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 line</td>
<td>1</td>
</tr>
<tr>
<td>pair of skew lines</td>
<td>$1 - \frac{1}{q} + \frac{1}{q^4}$</td>
</tr>
<tr>
<td>pair of intersecting lines</td>
<td>1</td>
</tr>
<tr>
<td>2 (or 3) intersecting lines</td>
<td>$1 - \frac{1}{q^3} + \frac{1}{q^4}$</td>
</tr>
<tr>
<td>“tritangent”</td>
<td>1</td>
</tr>
<tr>
<td>sextet of skew lines</td>
<td>$1 - \frac{1}{q} + \frac{1}{q^4}$</td>
</tr>
<tr>
<td>“double six”</td>
<td>$1 - \frac{1}{q}$</td>
</tr>
<tr>
<td>27 lines</td>
<td>$1 - \frac{15}{q} + \frac{81}{q^2} - \frac{185}{q^3} + \frac{150}{q^4}$</td>
</tr>
</tbody>
</table>

†using results from Bergvall–Gounelas ’19
Embrace the singularity

- \( M = (\mathbb{C}^{20} \setminus \Sigma) / \mathbb{C}^\times \), where
  \( \Sigma = \{ \text{singular cubic polynomials} \} \), the \textit{discriminant locus}
Embrace the singularity

- $M = (\mathbb{C}^{20} \setminus \Sigma)/\mathbb{C}^\times$, where 
  $\Sigma = \{\text{singular cubic polynomials}\}$, the discriminant locus

- Alexander duality $\implies H^*(M) \leftrightarrow H_*(\Sigma)$
Embrace the singularity

- \( M = (\mathbb{C}^{20} \setminus \Sigma) / \mathbb{C}^\times \), where
  \( \Sigma = \{ \text{singular cubic polynomials} \} \), the discriminant locus

- Alexander duality \( \implies H^*(M) \leftrightarrow H_*(\Sigma) \)

- Break up (stratify) \( \Sigma \) based on where \( F \in \Sigma \) is singular
Singular sets of singular cubics

all of $\mathbb{C}P^3$

three intersecting lines

two intersecting lines

four non-coplanar points

a line

two points

two points

a plane

a smooth conic and a point off its plane

a smooth conic

three non-collinear points

three non-collinear points

a point
Resolving $\Sigma$

- Replace $\Sigma$ by the simplicial resolution $\Sigma' \to \Sigma$ for ‘better’ pieces
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- Pieces of $\Sigma'$ are built out of:
  - a line
  - two points
  - a point

\[ \vdots \]
Resolving $\Sigma$

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- Pieces of $\Sigma'$ are built out of:
  - the vertices of the graph: e.g. space of all two point sets in $\mathbb{C}P^3$
Resolving $\Sigma$

- Replace $\Sigma$ by the simplicial resolution $\Sigma' \to \Sigma$ for ‘better’ pieces

- Pieces of $\Sigma'$ are built out of:
  - the vertices of the graph: e.g. space of all two point sets in $\mathbb{C}P^3$
  - the chains in the graph: e.g. space of all chains point $\subset$ two points $\subset$ line
Resolving $\Sigma$

- Replace $\Sigma$ by the simplicial resolution $\Sigma' \to \Sigma$ for ‘better’ pieces

\[ \cdots \]

\[ \begin{align*}
\text{a line} \\
\text{two points} \\
\text{a point}
\end{align*} \]

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  - the vertices of the graph: e.g. space of all two point sets in $\mathbb{C}P^3$
  - the chains in the graph: e.g. space of all chains point $\subset$ two points $\subset$ line

- Combine (co)homology of all the pieces in a spectral sequence . . .
### Keeping track of the lines and points

<table>
<thead>
<tr>
<th>Just the surface</th>
<th>With line $L$</th>
<th>With point $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>two points</strong></td>
<td>two points on $L$</td>
<td>$p$ and another point</td>
</tr>
<tr>
<td>11 pieces</td>
<td>one on $L$, one off $L$</td>
<td>two points collinear with $p$</td>
</tr>
<tr>
<td></td>
<td>both off $L$, coplanar with $L$</td>
<td>two points not collinear with $p$</td>
</tr>
<tr>
<td></td>
<td>two points not coplanar with $L$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>36 pieces</td>
<td>32 pieces</td>
</tr>
</tbody>
</table>
Singular sets of singular cubics containing the line $L$

- The line $L$ and two other lines concurrent with $L$, but not coplanar
- Three lines concurrent with $L$, with two of those coplanar with $L$
- Three concurrent lines with two coplanar with $L$
- A plane containing $L$, a plane not containing $L$
- A smooth conic tangent to $L$ and a point off its plane
- A smooth conic coplanar with $L$ and a point off its plane
- A smooth conic intersecting with $L$ but not coplanar and a point on $L$
- Three lines concurrent with $L$ but not coplanar with $L$
- Two points on $L$, one off $L$
- One point on $L$, two more coplanar with $L$
- Three points coplanar with $L$ but not on $L$
- Three points no two of which are coplanar with $L$
- Two points coplanar with $L$ but not on $L$
- Three points not coplanar with $L$
- Two points not coplanar with $L$
- One point on $L$, two more points coplanar with $L$ and another point
- Three points coplanar with $L$ and another point
- Two pairs of points separately coplanar with $L$
- A smooth conic coplanar with $L$, a smooth conic tangent to $L$
- A smooth conic tangent to $L$ and a point off its plane
- A smooth conic coplanar with $L$, a point not on $L$
- Two lines intersecting at a point of $L$
- Two lines not concurrent
- Two lines coplanar with $L$, but one intersecting $L$
- Two points on $L$, two other points
- One point on $L$, two more points coplanar with $L$, no two of those coplanar with $L$
- Three points coplanar with $L$, and another point
- Two points on $L$, one off $L$
- One point on $L$, two more not coplanar with $L$
- Three points coplanar with $L$ but not on $L$
- Two points not coplanar with $L$
- One point on $L$, one point off $L$
- Two points coplanar with $L$, one point off $L$
- Two points not coplanar with $L$
- A point on $L$, a point not on $L$
- Two points on $L$, one point off $L$
- Two points on $L$, one point not off $L$
- A line intersecting $L$
- A line skew with $L$
- Two points on $L$, one point off $L$
- One point on $L$, two more coplanar with $L$
- Three points coplanar with $L$ but not on $L$
- Two points coplanar with $L$, one point off $L$
- A line not containing $L$
- Two points on $L$, one point off $L$
- Two points coplanar with $L$ but not on $L$
- A point on $L$, a point not on $L$